Sec. 13.1:
Vector Functions and Space Curves

What We Will Go Over In Section 13.1

0. What is a Function?
1. What is a Vector (Valued) Functions
2. Domain of a Vector Function
3. Limits of Vector Functions
4. Continuity of Vector Functions
5. Tricks for Sketching Vector Functions / Tricks for Finding the Formula of a Vector Function

1. What is a vector valued function?

A vector valued function is a function whose inputs are real numbers and whose outputs are position vectors

Ex 1: Let $\vec{r}(t) =< t^2, 5 - t,\ t + 1 >$. Find…

a) $\vec{r}(1)$

b) $\vec{r}(4)$

1. What is a vector valued function?

A vector valued function is a function whose inputs are real numbers and whose outputs are position vectors

Notes:

- The domain of a vector function is a subset of $\mathbb{R}$.
- The codomain of a vector function is a subset of V_3 (or sometimes V_2).
- Each output vector is a position vector, so it is drawn with its tail at the origin.
- The head of each output vector is a point and the collection of these points is a curve (called a space curve).
- Vector functions are what we use to describe curves in $\mathbb{R}^3$.

1. What is a vector valued function?

A vector valued function is a function whose inputs are real numbers and whose outputs are position vectors

Notes:

- Often we think of the input as time and use the letter t for the input variable.
- The output vector is usually thought of as the position of an object at the time that was plugged in. Since the output is a position, we use the letter $\vec{r}$ for the output.
- The space curve is then the path that the object is moving.
- As t increases, we get a direction on the curve which indicates the direction the object is moving.

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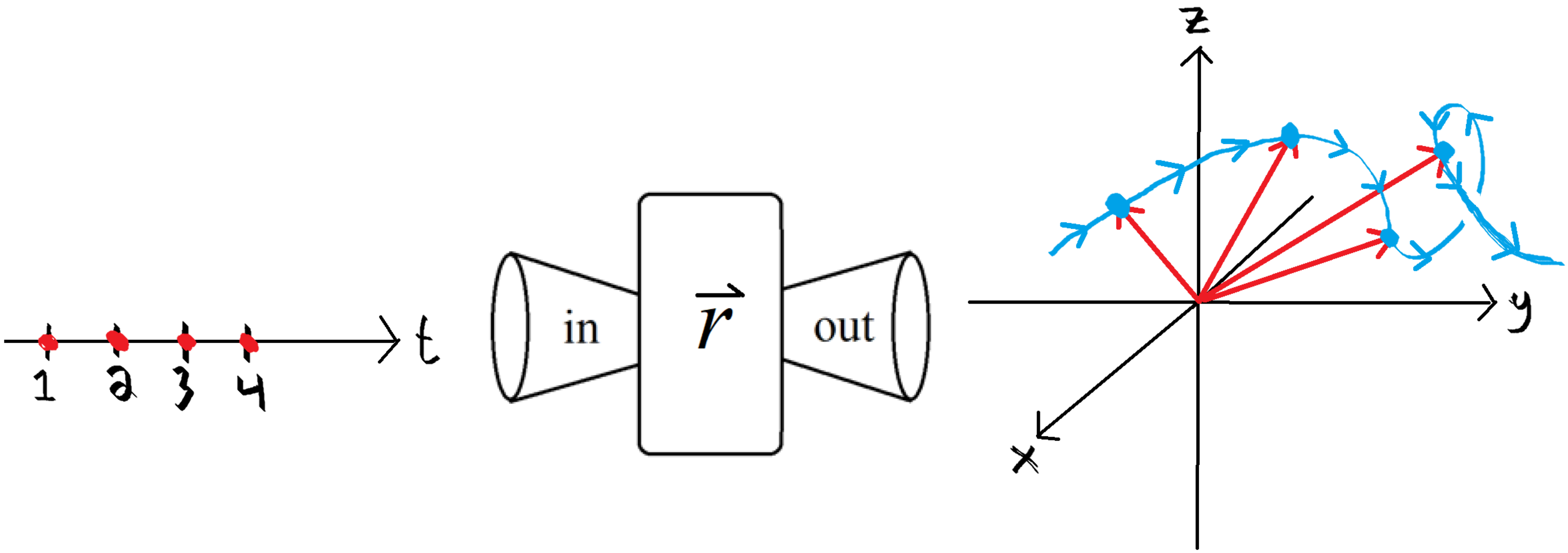

1. What is a vector valued function?

Ex 2: Determine and graph the space curve given by

a) $\vec{r}(t) =< 2 - 3t\ ,\ t\ ,\ 2t >$

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b) $\vec{r}(t) =< \sin(3t)\ ,\ t\ ,\ \cos(3t) >$

1. What is a vector valued function?

Ex 2: Determine and graph the space curve given by
c) What happens if the middle term changes to something like t^2 or $\frac{1}{t}$ in $\vec{r}(t) =< \sin(3t)\ ,\ t\ ,\ \cos(3t) >$

2. Domain of a vector function?

The <u>domain</u> of a function is the group of all possible inputs of a function.

<u>Notes</u>:

- The domain of a vector function is a subset of $\mathbb{R}$ (sometimes it's all of $\mathbb{R}$).
- Sometimes the domain of a function is given to you. In this case, nothing outside this given domain as an allowed input!
- If the domain of a vector function is not given to you, then the domain is all real numbers that that make ALL component functions defined.

2. Domain of a vector function?

Ex 3: Find the domain of the vector functions below.

a) $\vec{r}(t) =< 2 - 3t\ ,\ t\ ,\ 2t >\ ,\ 0 < t \leq 4$

2. Domain of a vector function?

Ex 3: Find the domain of the vector functions below.

b) $\vec{r}(t) = < \ln(7-t)\ ,\ \frac{1}{t-4}\ ,\ e^{t^2} >$

3. Limits of a vector function?

The limit of a vector function is defined by taking the limit of its component functions.

Definition:

Let $a \in \mathbb{R}$, and let $\vec{r}(t) = < f(t)\,,\ g(t)\,,\ h(t) >$.

Then

$$\lim_{t\to a} \vec{r}(t) \equiv < \lim_{t\to a} f(t)\,, \lim_{t\to a} g(t)\,, \lim_{t\to a} h(t) >$$

provided the limit of each individual component function exists.

3. Limits of a vector function?

Definition:

$$\lim_{t\to a} \vec{r}(t) \equiv < \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) >$$

Notes:

- a does not have to be in the domain of $\vec{r}(t)$.
- The limit only exists if the limit of ALL component functions exists. Otherwise the limit doesn't exist.
- If the limit of a component function is $\pm\infty$, the limit doesn't exist.
- The limit of a vector function is a position vector.

3. Limits of a vector function?

Ex 4: Find $\lim\limits_{t\to 0} \vec{r}(t)$ if $\vec{r}(t) = < \dfrac{\sin(t)}{t}\,,\ t^2\,,\ \dfrac{\sqrt{4+t}-2}{t} >$

4. Continuity of a vector function?

Definition:

Let $a \in \mathbb{R}$, and let $\vec{r}(t) =< f(t)\,,\ g(t)\,,\ h(t) >$.

Then $\vec{r}$ is continuous at $t = a$ if

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a).$$

Notes:

- $\vec{r}$ being continuous at $t = a$ actually means 3 things...

1) $\vec{r}$ is defined at $t = a$

   (that is, $\vec{r}(a)$ is defined)

2) $\lim_{t \to a} \vec{r}(t)$ exists

3) The answer from part 1 and part 2 are the same

   (that is $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$)

4. Continuity of a vector function?

<u>Definition</u>:

Let $a \in \mathbb{R}$, and let $\vec{r}(t) =< f(t)\,,\ g(t)\,,\ h(t) >$.

Then $\vec{r}$ is <u>continuous</u> at $t = a$ if

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a).$$

<u>Notes</u>:

- You can think of $\vec{r}$ being continuous at $t = a$ as meaning that you can draw the space curve through the time where $t = a$ without having to skip over the point where $t = a$ (fill the hole).
- $\vec{r}$ is continuous at $t = a$ if each component function is continuous at $t = a$.

4. Continuity of a vector function?

Ex 5: Is $\vec{r}(t) =< \cos(t)\,,\ 3t^2 + 9t - 7\,,\ \frac{1}{t+4} >$ continuous at $t = 0$?

5. Tricks for Sketching Vector Functions / Tricks for Finding the Formula of a Vector Function

Graphing space curves by hand is not always the easiest by hand. Technology can help, but even then sometimes you don’t get a good visual for what the curve looks like.

Tricks:

- Project the curve onto 1 of the coordinate planes
- Look at the curve from different vantage points
- See the curve as being contained in a surface that you know (like a sphere, cylinder, or quartic surface)

5. Tricks for Sketching Vector Functions / Tricks for Finding the Formula of a Vector Function

Ex 6: Find a vector and parametric equation for the line segment that starts at $P = (-1, 5, 3)$ and ends at $Q = (0, 4, -2)$.

5. Tricks for Sketching Vector Functions / Tricks for Finding the Formula of a Vector Function

Ex 7 (book example 6): Find the vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

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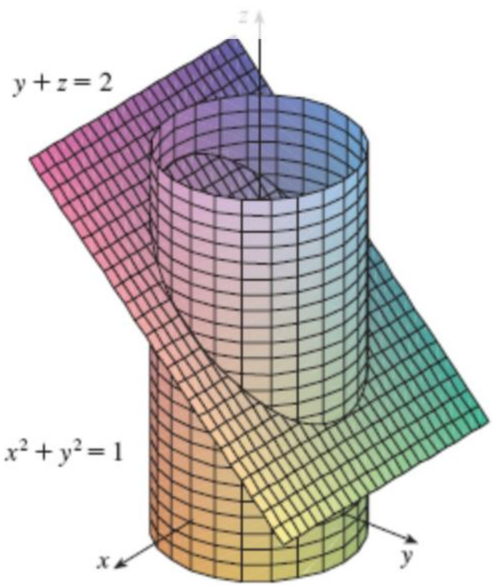
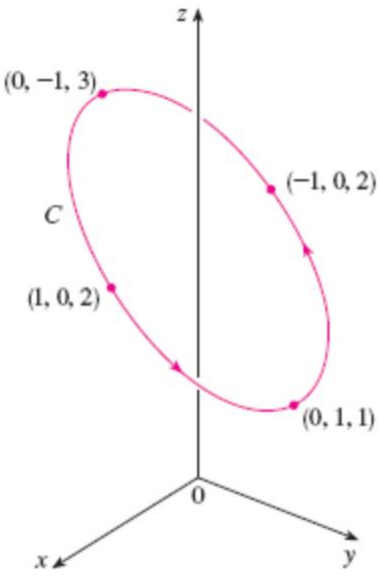
Ex 7 (book example 6): Find the vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

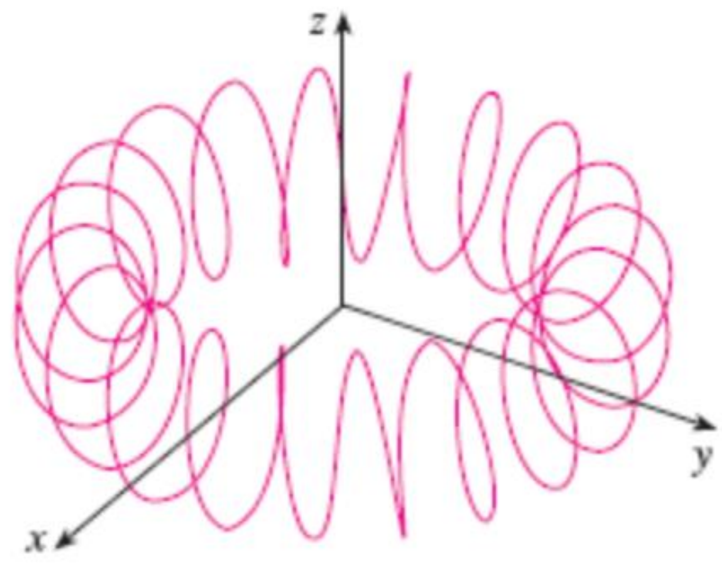




5. Tricks for Sketching Vector Functions / Tricks for Finding the Formula of a Vector Function

Toroidal Spiral

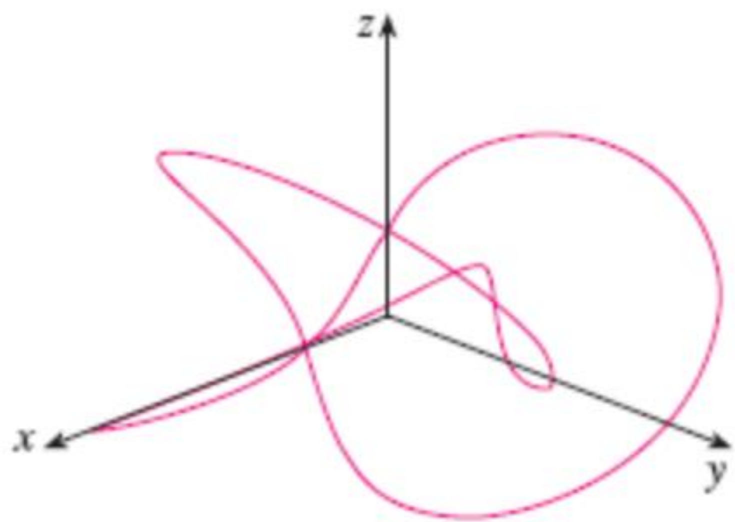
$$x = (4 + \sin 20t)\cos t$$
$$y = (4 + \sin 20t)\sin t$$
$$z = \cos 20t$$



5. Tricks for Sketching Vector Functions / Tricks for Finding the Formula of a Vector Function

Trefoil Knot

$$x = (2 + \cos 1.5t)\cos t$$
$$y = (2 + \cos 1.5t)\sin t$$
$$z = \sin 1.5t$$



5. Tricks for Sketching Vector Functions / Tricks for Finding the Formula of a Vector Function

Twisted Cubic

$$\vec{r}(t) = < t , t^2 , t^3 >$$

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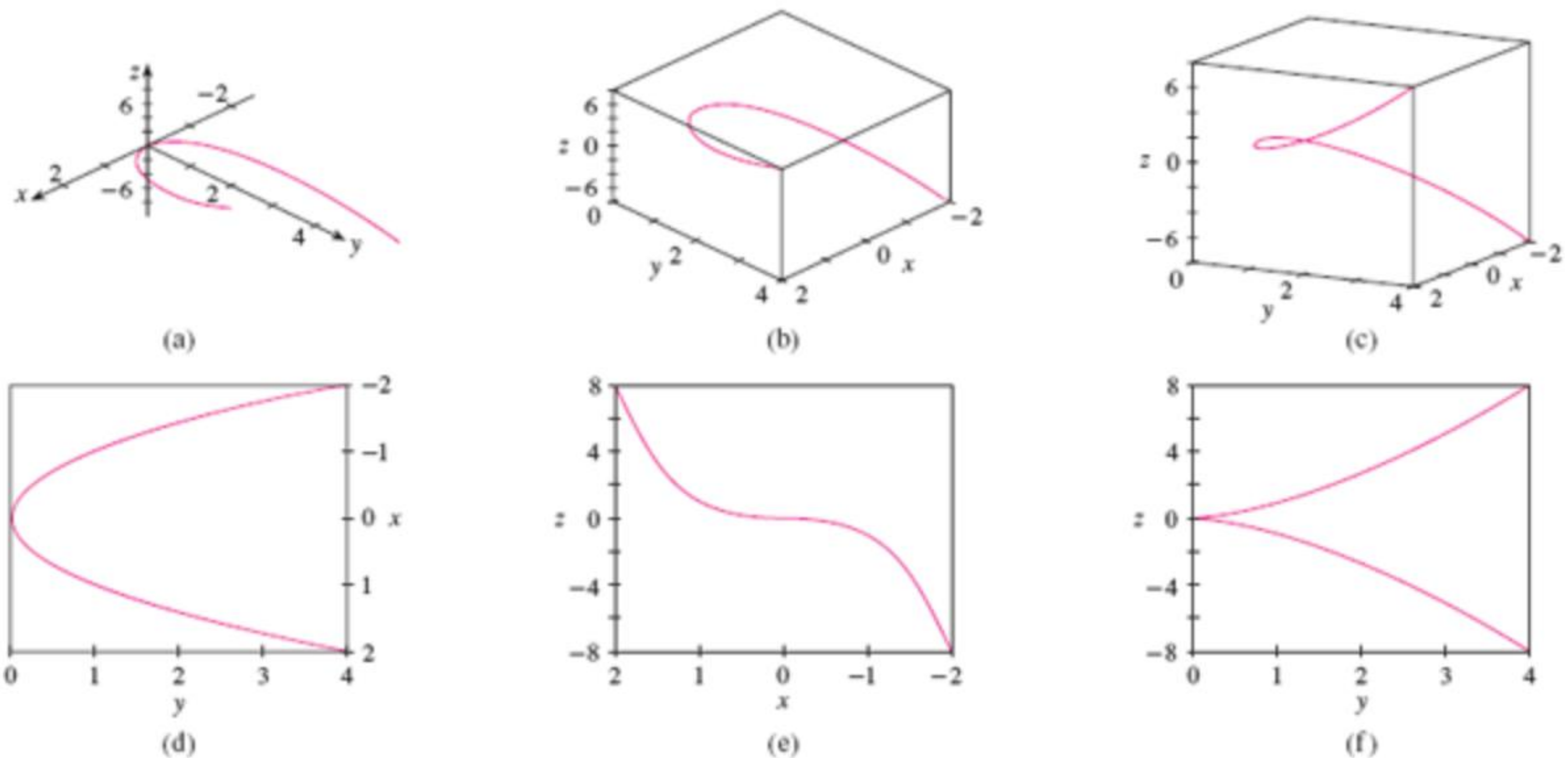

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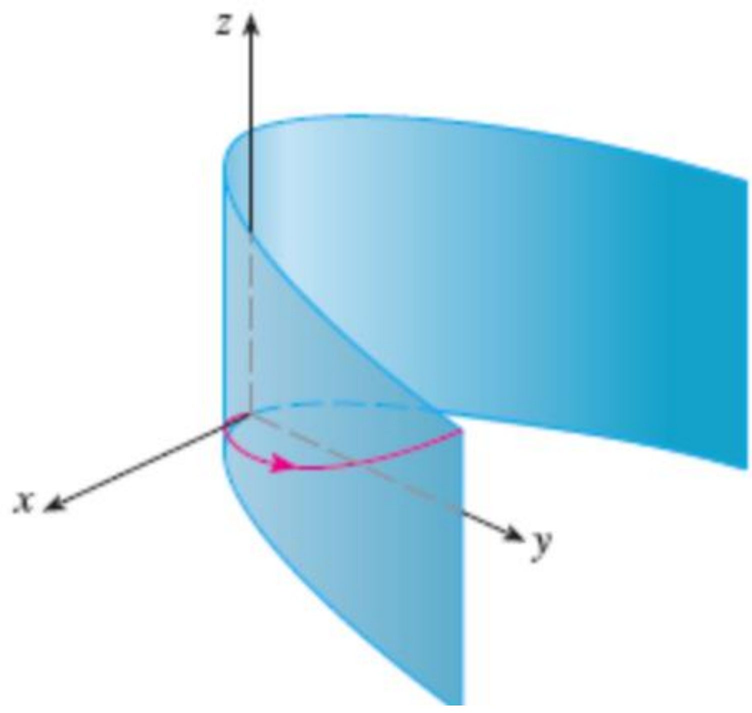


5. Tricks for Sketching Vector Functions / Tricks for Finding the Formula of a Vector Function

Some Homework Problems

7, 8, 9, 10, 11, 12, 13 and 14 Sketch the curve with the given vector equation. Indicate with an arrow the direction in which t increases.

7. $\mathbf{r}(t) = \langle \sin t, t \rangle$

8. $\mathbf{r}(t) = \langle t^2 - 1, t \rangle$

9. $\mathbf{r}(t) = \langle t, 2 - t, 2t \rangle$

10. $\mathbf{r}(t) = \langle \sin \pi t, t, \cos \pi t \rangle$

11. $\mathbf{r}(t) = \langle 3, t, 2 - t^2 \rangle$

12. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$

13. $\mathbf{r}(t) = t^2\mathbf{i} + t^4\mathbf{j} + t^6\mathbf{k}$

14. $\mathbf{r}(t) = \cos t\,\mathbf{i} - \cos t\,\mathbf{j} + \sin t\,\mathbf{k}$

5. Tricks for Sketching Vector Functions / Tricks for Finding the Formula of a Vector Function

Some Homework Problems

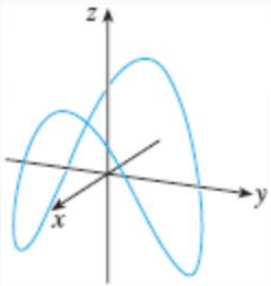
15 and 16 Draw the projections of the curve on the three coordinate planes. Use these projections to help sketch the curve.

15. $\mathbf{r}(t) = \langle t,\ \sin t, 2\cos t \rangle$
16. $\mathbf{r}(t) = \langle t, t, t^2 \rangle$

5. Tricks for Sketching Vector Functions / Tricks for Finding the Formula of a Vector Function

Some Homework Problems

I.



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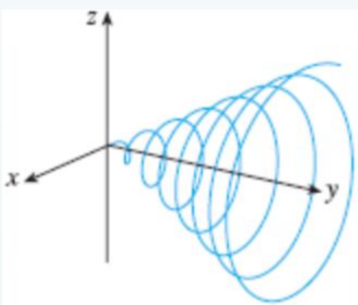

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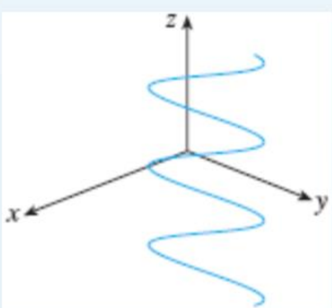

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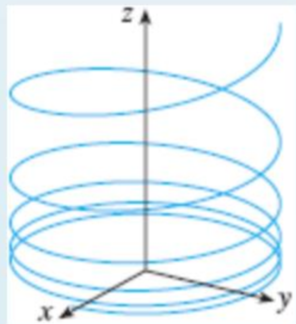
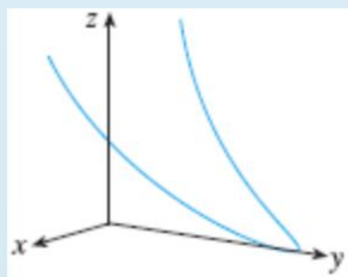
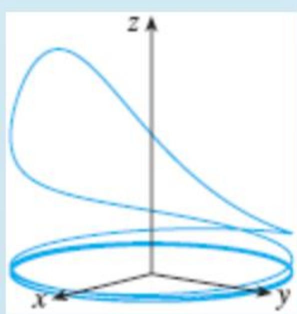


V.



VI.



21. $x = t\cos t, \quad y = t, \quad z = t\sin t, \quad t \geqslant 0$

22. $x = \cos t, \quad y = \sin t, \quad z = 1/(1+t^2)$

23. $x = t, \quad y = 1/(1+t^2), \quad z = t^2$

24. $x = \cos t, \quad y = \sin t, \quad z = \cos 2t$

25. $x = \cos 8t, \quad y = \sin 8t, \quad z = e^{0.8t}, \quad t \geqslant 0$

26. $x = \cos^2 t, \quad y = \sin^2 t, \quad z = t$

5. Tricks for Sketching Vector Functions / Tricks for Finding the Formula of a Vector Function

Some Homework Problems

27. Show that the curve with parametric equations $x = t\cos t$, $y = t\sin t$, $z = t$ lies on the cone $z^2 = x^2 + y^2$, and use this fact to help sketch the curve.

28. Show that the curve with parametric equations $x = \sin t$, $y = \cos t$, $z = \sin^2 t$ is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$. Use this fact to help sketch the curve.

29. Find three different surfaces that contain the curve $\mathbf{r}(t) = 2t\,\mathbf{i} + e^t\,\mathbf{j} + e^{2t}\,\mathbf{k}$.

30. Find three different surfaces that contain the curve $\mathbf{r}(t) = t^2\,\mathbf{i} + \ln t\,\mathbf{j} + (1/t)\,\mathbf{k}$.

31. At what points does the curve $\mathbf{r}(t) = t\,\mathbf{i} + (2t - t^2)\,\mathbf{k}$ intersect the paraboloid $z = x^2 + y^2$?

32. At what points does the helix $\mathbf{r}(t) = \langle \sin t,\ \cos t, t\rangle$ intersect the sphere $x^2 + y^2 + z^2 = 5$?

5. Tricks for Sketching Vector Functions / Tricks for Finding the Formula of a Vector Function

Some Homework Problems

42, 43, 44, 45 and 46 Find a vector function that represents the curve of intersection of the two surfaces.

42. The cylinder $x^2 + y^2 = 4$ and the surface $z = xy$

43. The cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$

44. The paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$

45. The hyperboloid $z = x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$

46. The semiellipsoid $x^2 + y^2 + 4z^2 = 4$, $y \geqslant 0$, and the cylinder $x^2 + z^2 = 1$